Divisorial contractions in dimension 3 which contract divisors to smooth points

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Abstract

We deal with a divisorial contraction in dimension 3 which contracts its exceptional divisor to a smooth point. We prove that any such contraction can be obtained by a suitable weighted blow-up.

0 Introduction

Divisorial contractions play a major role in the minimal model program ([KMM87]). Now that we know this program works in dimension 3 ([M88]), it is desirable to describe them explicitly in dimension 3. Moreover also in view of the Sarkisov program ([Co95]) and its applications (for example [CPR99]), we can recognize the importance of such description since Sarkisov links of types I and II in this program start from the converse of divisorial contractions.

Now we concentrate on divisorial contractions in dimension 3. Let $f\colon (Y \supset E) \to (X \ni P)$ be such a contraction. There are two ways to deal with f, that is to say, one starting from Y, and the other from X. From the former standpoint, S. Mori classified them in the case when Y is smooth ([M82]), and S. Cutkosky extended this result to the case when Y has only terminal Gorenstein singularities ([Cu88]). On the other hand, from the latter standpoint, Y. Kawamata showed that f must be a certain weighted blow-up when P is a terminal quotient singularity ([K96]), and A. Corti showed that f must be the blow-up when P is an ordinary double point ([Co99, Theorem 3.10]).

While it seems that singularities on Y make it hard to tackle the problem in the former case, the singularity of P may be useful in the latter case because it gives a special filtration in the tangent space at P. In this paper we treat the case when P is a smooth point and prove the following theorem:

Theorem 1.2. *Let Y be a 3-dimensional $\mathbb{Q}$-factorial normal variety with only terminal singularities, and let $f\colon (Y \supset E) \to (X \ni P)$ be an algebraic germ of a divisorial contraction which contracts its exceptional divisor E to a smooth point P. Then we can take local parameters x, y, z at P and coprime positive integers a and b, such that f is the weighted blow-up of X with its weights $(x, y, z) = (1, a, b)$.*

Now we explain our approach to the problem. Y. Kawamata adopted the method of comparing discrepancies of exceptional divisors, and A. Corti applied Shokurov's connectedness lemma ([K$^+$92, Theorem 17.4]). But in the case when P is a smooth point, these methods do not work well if the center of E on $\mathrm{Bl}_P(X)$ is a point. Our main tools are the singular Riemann-Roch formula ([R87, Theorem 10.2]) on Y and a relative vanishing theorem ([KMM87, Theorem 1-2-5]) with respect to f. First with them we derive a rather simple formula for $\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE)$'s and an upper-bound of the number of fictitious non-Gorenstein points of Y (Proposition 2.7). Next using this upper-bound, we show that the coefficient of E in the pull-back of a general prime divisor through P is 1 (Subsection 2.3). And finally investigating the values of $\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE)$'s more carefully, we prove the theorem (Subsection 2.4).

I wish to express my gratitude to Professor Yujiro Kawamata for his valuable comments and warm encouragement. He also recommended me to read the papers [CPR99] and [Co99]. In fact I found the problem treated here as [Co99, Conjecture 3.11].

1 Statement of the theorem

We work over an algebraically closed field k of characteristic zero. A variety means an integral separated scheme of finite type over $\mathrm{Spec}\, k$. We use basic terminologies in [K$^+$92, Chapters 1, 2].

Before we state the theorem, we have to define a divisorial contraction. In this paper it means a morphism which may emerge in the minimal model program (see [KMM87]).

Definition 1.1. Let $f\colon Y \to X$ be a morphism with connected fibers between normal varieties. We call f a *divisorial contraction* if it satisfies the following conditions:

1. Y is $\mathbb{Q}$-factorial with only terminal singularities.
2. The exceptional locus of f is a prime divisor.
3. $-K_Y$ is f-ample.
4. The relative Picard number of f is 1.

Now it is the time when we state the theorem precisely.

Theorem 1.2. *Let Y be a 3-dimensional $\mathbb{Q}$-factorial normal variety with only terminal singularities, and let $f\colon (Y \supset E) \to (X \ni P)$ be an algebraic germ of a divisorial contraction which contracts its exceptional divisor E to a smooth point P. Then we can take local parameters x, y, z at P and coprime positive integers a and b, such that f is the weighted blow-up of X with its weights $(x, y, z) = (1, a, b)$.*

2 Proof of the theorem

2.1 Strategy for its proof

We may assume that X is projective and smooth, and consider its algebraic germ if necessary. First we construct a series of birational morphisms.

Construction 2.1. We construct birational morphisms $g_i\colon X_i \to X_{i-1}$ between smooth varieties, integral closed subschemes $Z_i \subset X_i$, and prime divisors F_i on X_i inductively, and define positive integers n, m, with the following procedure:

1. Define X_0 as X and Z_0 as P.

2. Let $b_i\colon \mathrm{Bl}_{Z_{i-1}}(X_{i-1}) \to X_{i-1}$ be the blow-up of X_{i-1} along Z_{i-1}, and let $b'_i\colon X_i \to \mathrm{Bl}_{Z_{i-1}}(X_{i-1})$ be a resolution of $\mathrm{Bl}_{Z_{i-1}}(X_{i-1})$, that is, a proper birational morphism from a smooth variety X_i which is isomorphic over the smooth locus of $\mathrm{Bl}_{Z_{i-1}}(X_{i-1})$. We note that b'_i is isomorphic at the generic point of the center of E on $\mathrm{Bl}_{Z_{i-1}}(X_{i-1})$. We define $g_i = b_i \circ b'_i\colon X_i \to X_{i-1}$.

3. Define Z_i as the center of E on X_i with the reduced induced closed subscheme structure, and F_i as the only g_i-exceptional prime divisor on X_i which contains Z_i.

4. We stop this process when $Z_n = F_n$. This process must terminate after finite steps (see Remark 2.1.2) and thus we get the sequence $X_n \to \cdots \to X_0$.

5. We define $m \leq n$ as the largest integer such that Z_{m-1} is a point.

6. We define $g_{ji}\,(j \leq i)$ as the morphism from X_i to X_j.

Remark 2.1.1. We remark that $f_*\mathcal{O}_Y(-iE) = g_{0n*}\mathcal{O}_{X_n}(-iF_n)$ for any i because E and F_n are the same as valuations.

Remark 2.1.2. We prove the termination of the process. Assume that we have the sequence $X_l \to \cdots \to X_0$ and $Z_l \neq F_l$. We take common resolutions of X_l and Y over X, that is, birational morphisms $h\colon W \to X_l$ and $h'\colon W \to Y$ from a smooth variety W such that $g_{0l} \circ h = f \circ h'$. We put

$$
\begin{aligned}
K_Y &= f^*K_X + aE,\\
K_{X_l} &= g_{0l}^*K_X + sF_l + (\text{others}),\\
K_W &= h^*K_{X_l} + c({h'}^{-1})_*E + (\text{others}),\\
h^*F_l &= (h^{-1})_*F_l + t({h'}^{-1})_*E + (\text{others}).
\end{aligned}
$$

We note that a, s, c and t are positive integers. Then

$$\begin{aligned}K_W &= h'^*(f^*K_X + aE) + (\text{others}) \\ &= h^*(g_{0l}^*K_X + sF_l + (\text{others})) + c({h'}^{-1})_*E + (\text{others}) \\ &= h^*g_{0l}^*K_X + s(h^{-1})_*F_l + (st+c)({h'}^{-1})_*E + (\text{others}).\end{aligned}$$

Comparing the coefficients of $({h'}^{-1})_*E$, we have $a = st + c$ and especially $a > s$. On the other hand because we know $s \geq l+1$ by the construction of F_l, we get $a > l+1$. It shows that the above process terminates with $n \leq a-1$. □

We state an easy lemma.

Lemma 2.2. *Let $f_i\colon (Y_i \supset E_i) \to (X \supset f_i(E_i))$ with $i = 1, 2$ be algebraic germs of divisorial contractions. Assume that E_1 and E_2 are the same as valuations. Then f_1 and f_2 are isomorphic as morphisms over X.*

Proof. Let $g_i\colon Z \to Y_i$ with $i = 1, 2$ be common resolutions and $h = f_i \circ g_i$. We choose g_i-exceptional effective $\mathbb{Q}$-divisors F_i ($i = 1, 2$) and a $\mathbb{Q}$-divisor G on Z such that $G = -g_1^*E_1 + F_1 = -g_2^*E_2 + F_2$. Then,

$$Y_i = \operatorname{Proj}_X \oplus_{j\geq 0} f_{i*}\mathcal{O}_{Y_i}(-jE_i) = \operatorname{Proj}_X \oplus_{j\geq 0} h_*\mathcal{O}_Z(jG).$$

□

For weighted blow-ups in dimension 3, we have a criterion on terminal singularities.

Theorem 2.3. *Let $X \ni P$ be an algebraic germ of a smooth 3-dimensional variety with local parameters x, y, z at P, let r, a, b be positive integers with $r \leq a \leq b$, and let $Y \to X$ be the weighted blow-up of X with its weights $(x, y, z) = (r, a, b)$. Then Y has only terminal singularities if and only if $r = 1$ and a, b are coprime.*

By the above lemma and theorem, the problem is reduced to proving that F_n equals, as valuations, an exceptional divisor obtained by a weighted blow-up of X. We restate this in terms of ideal sheaves of $\mathcal{O}_X$.

Proposition 2.4. *(Notation as above). F_n equals, as valuations, an exceptional divisor obtained by a weighted blow-up of X with its weights $(x, y, z) = (1, m, n)$ for suitable local parameters x, y, z at P, if and only if the following conditions hold:*

1. *$f_*\mathcal{O}_Y(-2E) \neq \mathfrak{m}_P$, that is, $g_{0n*}\mathcal{O}_{X_n}(-2F_n) \neq \mathfrak{m}_P$.*
2. *$f_*\mathcal{O}_Y(-nE) \not\subseteq \mathfrak{m}_P^2$, that is, $g_{0n*}\mathcal{O}_{X_n}(-nF_n) \not\subseteq \mathfrak{m}_P^2$.*

Here $\mathfrak{m}_P \subset \mathcal{O}_X$ is the ideal sheaf of P.

Proof. The "only if" part is obvious taking it into account that for any i $g_{0n*}\mathcal{O}_{X_n}(-iF_n) = (x^s y^t z^u | s + mt + nu \geq i)$. Actually $x \notin g_{0n*}\mathcal{O}_{X_n}(-2F_n)$ and $z \in g_{0n*}\mathcal{O}_{X_n}(-nF_n)$.

Now we prove the "if" part. The condition 1 means that the coefficient of F_n in $g_{1n}^* F_1$ is 1. This says that for any $i \geq 1$, F_i is the only g_{0i}-exceptional prime divisor on X_i containing Z_i and the coefficient of F_n in $g_{in}^* F_i$ is 1.

We consider a prime divisor $D \ni P$ on X which is smooth at P and define $1 \leq l \leq n$ as the largest integer such that $Z_{l-1} \subseteq (g_{0,l-1}^{-1})_* D$. Then $(g_{0i}^{-1})_* D$ is smooth at the generic point of Z_i for any $i < l$, and so we get $g_{0l}^* D = (g_{0l}^{-1})_* D + \sum_{i=1}^{l} i\,(g_{il}^{-1})_* F_i + (\text{others})$. Therefore the coefficient of F_n in $g_{0n}^* D$ is l. By the condition 2, we can choose $z \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ such that $g_{0n}^* \mathrm{div}(z) \geq nF_n$, that is, $Z_{n-1} \subseteq (g_{0,n-1}^{-1})_* \mathrm{div}(z)$ because of the above argument. Adding $x, y \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$ such that $Z_{m-1} \subseteq (g_{0,m-1}^{-1})_* \mathrm{div}(y)$, we can take local parameters x, y, z at P. Then $F_i\,(1 \leq i \leq n)$ equals, as valuations, the exceptional divisor obtained by the weighted blow-up of X with its weights $(x, y, z) = (1, \min\{i, m\}, i)$, and especially F_n is obtained by the weighted blow-up of X with its weights $(x, y, z) = (1, m, n)$. $\square$

So we prove the above two conditions.

2.2 Preliminaries

Let $K_Y = f^* K_X + aE$, and let r be the global Gorenstein index of Y, that is, the smallest positive integer such that rK_Y is Cartier. Since a equals the discrepancy of F_n with respect to K_X, $a \in \mathbb{Z}_{\geq 2}$.

Lemma 2.5. *(Notation as above). a and r are coprime.*

Proof. Let s be the greatest common divisor of a and r, and let $a = sa', r = sr'$. Since $r'aE = a'rE$ is Cartier by [K88, Corollary 5.2], so is $r'K_Y$. Hence $r' = r$ and $s = 1$. $\square$

We recall the singular Riemann-Roch formula ([R87, Theorem 10.2]).

Theorem 2.6. *Let X be a projective 3-dimensional variety with only canonical singularities, and let D be a Weil divisor on X such that for any $P \in X$ there exists an integer i_P satisfying $(\mathcal{O}_X(D))_P \cong (\mathcal{O}_X(i_P K_X))_P$. Then there is a formula of the form*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} D \cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over singular points of X, and $c_P(D) \in \mathbb{Q}$ is a contribution depending only on the local analytic type of $P \in X$ and $\mathcal{O}_X(D)$.

If P is a terminal quotient singularity of type $\frac{1}{r_P}(1,-1,b_P)$, then

$$c_P(D) = -\overline{i_P}\frac{r_P^2-1}{12r_P} + \sum_{j=1}^{\overline{i_P}-1} \frac{\overline{jb_P}(r_P - \overline{jb_P})}{2r_P},$$

where $^-$ *denotes the smallest residue modulo r_P, that is, $\overline{j} = j - \lfloor \frac{j}{r_P} \rfloor r_P$ in terms of the round down $\lfloor\ \rfloor$. The definition of the round down $\lfloor\ \rfloor$ is $\lfloor j \rfloor = \max\{k \in \mathbb{Z} | k \le j\}$.*

And for any terminal singularity P,

$$c_P(D) = \sum_\alpha c_{P_\alpha}(D_\alpha),$$

where $\{(P_\alpha, D_\alpha)\}_\alpha$ is a flat deformation of (P,D) to terminal quotient singularities.

Remark 2.6.1. If X has only terminal singularities, then we can write the contribution term $\sum_P c_P(D)$ as $\sum_Q c_Q(D)$, where

$$c_Q(D) = -\overline{i_Q}\frac{r_Q^2-1}{12r_Q} + \sum_{j=1}^{\overline{i_Q}-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}.$$

For its summation takes place over points which need not lie on X but may lie on deformed varieties of X, Q's are called "fictitious" points in the sense of M. Reid. This description holds even though X has canonical singularities, but in this case Q's may lie on deformed varieties of crepant blown-up varieties of X (see [R87] for details).

By Lemma 2.5, we can take an integer e such that $ae \equiv 1$ modulo r. Then $(\mathcal{O}_Y(E))_Q \cong (\mathcal{O}_Y(eK_Y))_Q$ for any $Q \in E$. Using the singular Riemann-Roch formula, we get

$$\begin{aligned}\chi(\mathcal{O}_Y(iE)) = \chi(\mathcal{O}_Y) &+ \frac{1}{12}i(i-a)(2i-a)E^3 \\ &+ \frac{1}{12}iE\cdot c_2(Y) + A_i,\end{aligned} \tag{2.1}$$

where A_i is the contribution term and has the below description:

$$A_i = \sum_{Q\in I} c_Q(iE),$$

$$c_Q(iE) = -\overline{ie}\frac{r_Q^2-1}{12r_Q} + \sum_{j=1}^{\overline{ie}-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}.$$

Here $Q \in I$ are fictitious singularities. The type of Q is $\frac{1}{r_Q}(1,-1,b_Q)$, $(\mathcal{O}_{Y_Q}(E_Q))_Q \cong (\mathcal{O}_{Y_Q}(eK_{Y_Q}))_Q$ where (Y_Q, E_Q) is the fictitious pair for Q,

and $\bar{}$ denotes the smallest residue modulo r_Q. We note that b_Q is coprime to r_Q and also e is coprime to r_Q because $r|(ae-1)$. So $v_Q = \overline{eb_Q}$ is coprime to r_Q. With this description, $r = 1$ if I is empty, and otherwise r is the lowest common multiple of $\{r_Q\}_{Q\in I}$. We note that $c_Q(iE)$ depends only on i mod r_Q and equals 0 if $r_Q|i$. Especially A_i depends only on i mod r and equals 0 if $r|i$.

We put $B_i = -(A_i + A_{-i})$. Because

$$
\begin{aligned}
c_Q(iE) + c_Q(-iE) &= \Big(-\overline{ie}\frac{r_Q^2-1}{12r_Q} + \sum_{j=1}^{\overline{ie}-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}\Big) \\
&\quad + \Big(-\overline{-ie}\frac{r_Q^2-1}{12r_Q} + \sum_{j=1}^{\overline{-ie}-1} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}\Big) \\
&= -\frac{r_Q^2-1}{12} + \Big(\sum_{j=1}^{r_Q} \frac{\overline{jb_Q}(r_Q - \overline{jb_Q})}{2r_Q}\Big) - \frac{\overline{ieb_Q}(r_Q - \overline{ieb_Q})}{2r_Q} \\
&= -\frac{r_Q^2-1}{12} + \Big(\sum_{j=1}^{r_Q} \frac{j(r_Q - j)}{2r_Q}\Big) - \frac{\overline{iv_Q}(r_Q - \overline{iv_Q})}{2r_Q} \\
&= -\frac{\overline{iv_Q}(r_Q - \overline{iv_Q})}{2r_Q}
\end{aligned}
$$

where the third equality comes from the property that b_Q and r_Q are coprime, we have

$$
B_i = -\sum_{Q\in I}(c_Q(iE) + c_Q(-iE)) = \sum_{Q\in I} \frac{\overline{iv_Q}(r_Q - \overline{iv_Q})}{2r_Q}. \tag{2.2}
$$

Proposition 2.7. *(Notation as above).*

(A) $rE^3 \in \mathbb{Z}_{>0}$.

(B) $1 = \frac{1}{2}aE^3 + \sum_{Q\in I} \frac{v_Q(r_Q - v_Q)}{2r_Q}$.

(C) $\dim_k \mathcal{O}_X / f_*\mathcal{O}_Y(-iE) =$

$$
i^2 - \frac{1}{2}\sum_{Q\in I} \min_{0\le j<i}\{(1+j)jr_Q + i(i-1-2j)v_Q\} \quad (1 \le i \le a).
$$

(D) $\sum_{Q\in I} \min\{v_Q, r_Q - v_Q\} = \dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2$.

Remark 2.7.1. In particular (A), (C) and (D) are essential. We use (A) to bound the value of a from above and use (C) to control the values of r_Q's.

(D) shows that the number of fictitious non-Gorenstein points of Y is at most 3. We prove the conditions 1 and 2 in Proposition 2.4 according to the value of $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2$.

Remark 2.7.2. In fact, because of (2.2) and (2.9) the right hand side of (C) is the same if we replace v_Q by $r_Q - v_Q$.

Proof. We consider the exact sequence:

$$0 \to \mathcal{O}_Y((i-1)E) \to \mathcal{O}_Y(iE) \to \mathcal{Q}_i \to 0. \tag{2.3}$$

By (2.1), we get

$$\begin{aligned} \chi(\mathcal{Q}_i) &= \chi(\mathcal{O}_Y(iE)) - \chi(\mathcal{O}_Y((i-1)E)) \\ &= \frac{1}{12}\{2(3i^2 - 3i + 1) - 3(2i-1)a + a^2\}E^3 \\ &\quad + \frac{1}{12}E \cdot c_2(Y) + A_i - A_{i-1}. \end{aligned} \tag{2.4}$$

Since $\chi(\mathcal{Q}_i) - \chi(\mathcal{Q}_{r+i}) = \frac{r}{2}(a+1-r-2i)E^3$ is an integer for any i and E^3 is positive, we have (A).

By (2.4),

$$\chi(\mathcal{Q}_{-i}) - \chi(\mathcal{Q}_{i+1}) = (i + \frac{1}{2})aE^3 + B_{i+1} - B_i. \tag{2.5}$$

Let $d(i) = \dim_k f_*\mathcal{O}_Y(iE)/f_*\mathcal{O}_Y((i-1)E)$. We note that $d(i) = 0$ if $i \geq 1$, and $d(0) = 1$. Because $(Y, \varepsilon E)$ is weak KLT and $iE - (K_Y + \varepsilon E)$ is f-ample for a sufficiently small positive rational number ε and an integer $i \leq a$, using [KMM87, Theorem 1-2-5], we have $R^j f_*\mathcal{O}_Y(iE) = 0$ for $i \leq a$, $j \geq 1$. So by (2.3), for any $i \leq a$,

$$H^0(Y, \mathcal{Q}_i) = f_*\mathcal{Q}_i = f_*\mathcal{O}_Y(iE)/f_*\mathcal{O}_Y((i-1)E),$$

$$H^j(Y, \mathcal{Q}_i) = R^j f_*\mathcal{Q}_i = 0 \quad \text{for} \quad j \geq 1,$$

and therefore $d(i) = \chi(\mathcal{Q}_i)$.

Putting $i = 0$ in (2.5), we get

$$1 = \frac{1}{2}aE^3 + B_1. \tag{2.6}$$

Combining this and (2.2) with $i = 1$, we get (B).

With (2.5), we obtain for $1 \leq i \leq a$,

$$\begin{aligned} \sum_{1 \leq j < i} d(-j) &= \sum_{1 \leq j < i} \{\chi(\mathcal{Q}_{-j}) - \chi(\mathcal{Q}_{j+1})\} \\ &= \sum_{1 \leq j < i} \{(j + \frac{1}{2})aE^3 + B_{j+1} - B_j\} \\ &= \frac{1}{2}(i^2 - 1)aE^3 + B_i - B_1. \end{aligned} \tag{2.7}$$

Eliminating $\frac{1}{2}aE^3$ with (2.6), we obtain

$$\sum_{1\le j<i} d(-j) = (i^2-1) + B_i - i^2 B_1 \quad (1\le i\le a). \tag{2.8}$$

Since for $i \ge 1$,

$$\begin{aligned}
&\frac{\overline{iv_Q}(r_Q-\overline{iv_Q})}{2r_Q} - i^2\frac{v_Q(r_Q-v_Q)}{2r_Q}\\
&= -\frac{1}{2}\left\{ r_Q\Big(\frac{iv_Q-\overline{iv_Q}}{r_Q} - \frac{iv_Q}{r_Q}+\frac{1}{2}\Big)^2 + i^2\frac{v_Q(r_Q-v_Q)}{r_Q} - \frac{r_Q}{4} \right\}\\
&= -\frac{1}{2}\left\{ r_Q\Big(\Big\lfloor\frac{iv_Q}{r_Q}\Big\rfloor - \frac{iv_Q}{r_Q}+\frac{1}{2}\Big)^2 + i^2\frac{v_Q(r_Q-v_Q)}{r_Q} - \frac{r_Q}{4} \right\}\\
&= -\frac{1}{2}\min_{0\le j<i}\left\{ r_Q\Big(j - \frac{iv_Q}{r_Q}+\frac{1}{2}\Big)^2 + i^2\frac{v_Q(r_Q-v_Q)}{r_Q} - \frac{r_Q}{4} \right\}\\
&= -\frac{1}{2}\min_{0\le j<i}\{(1+j)jr_Q + i(i-1-2j)v_Q\},
\end{aligned}$$

with (2.2) we have

$$B_i - i^2B_1 = -\frac{1}{2}\sum_{Q\in I}\min_{0\le j<i}\{(1+j)jr_Q + i(i-1-2j)v_Q\} \quad (i\ge 1). \tag{2.9}$$

Of course because $\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) = 1+\sum_{j=1}^{i-1} d(-j)$, combining this with (2.8) and (2.9), we obtain (C).

Putting $i=2$ in (2.8) and (2.9), we have

$$d(-1) = 3 - \sum_{Q\in I}\min\{v_Q, r_Q-v_Q\}.$$

Since $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2 = 3-d(-1)$, we get (D). $\square$

2.3 Proof of $\mathbf{f_*\mathcal{O}_Y(-2E)\neq \mathfrak{m}_P}$

Assuming that $f_*\mathcal{O}_Y(-2E) = \mathfrak{m}_P$, we will derive a contradiction. The assumption means that the coefficient of F_n in $g_{1n}^*F_1$ is bigger than 1, so there exists a Z_i which is contained in at least two g_{0i}-exceptional prime divisors on X_i. The minimum value of a in this case occurs when Z_1 is a curve, $Z_2 = (g_2^{-1})_*F_1\cap F_2$, and $n=3$, and the minimum value is 6. So we get $a\ge 6$. By the assumption and (D), we obtain $\sum_{Q\in I}\min\{v_Q, r_Q-v_Q\} = 3$. Thus we have only to consider the three cases:

Case 1. $\{(r_Q,\overline{v_Q})\}_{Q\in I} = \{(r,\overline{\pm 3})\},\ r\ge 7.$
Case 2. $\{(r_Q,\overline{v_Q})\}_{Q\in I} = \{(r_1,\overline{\pm 1}),(r_2,\overline{\pm 2})\},\ r_1\ge 2,\ r_2\ge 5.$
Case 3. $\{(r_Q,\overline{v_Q})\}_{Q\in I} = \{(r_1,\overline{\pm 1}),(r_2,\overline{\pm 1}),(r_3,\overline{\pm 1})\},\ 2\le r_1\le r_2\le r_3.$

Here $\pm$ means that one of these occurs for each $\overline{v_Q}$. We remark that v_Q is coprime to r_Q.

Since $\displaystyle\sum_{Q\in I}\frac{v_Q(r_Q-v_Q)}{2r_Q}<1$ from (B), we have the below inequalities:

Case 1. $3/2-9/2r<1$.
Case 2. $3/2-(1/2r_1+2/r_2)<1$.
Case 3. $3/2-(1/2r_1+1/2r_2+1/2r_3)<1$.

Using this evaluation, we can restrict possible values of r_Q's. Below we show all the possible values and the corresponding values of aE^3:

Case 1.

r	:	7	8
aE^3	:	2/7	1/8

Case 2.

(r_1,r_2)	:	$(2,5)$	$(3,5)$	$(4,5)$	$(2,7)$
aE^3	:	3/10	2/15	1/20	1/14

Case 3.

(r_1,r_2,r_3)	:	$(2,2,r_3)$	$(2,3,3)$	$(2,3,4)$	$(2,3,5)$
aE^3	:	$2/2r_3$	1/6	1/12	1/30

Recalling that r is the lowest common multiple of $\{r_Q\}_{Q\in I}$, with (A) we have $a\leq 3$ for all the above cases. This contradicts $a\geq 6$. $\square$

2.4 Proof of $\mathbf{f}_*\mathcal{O}_{\mathbf{Y}}(-\mathbf{nE})\not\subseteq\mathfrak{m}_{\mathbf{P}}^{\mathbf{2}}$

Because $f_*\mathcal{O}_Y(-2E)\neq\mathfrak{m}_P$, we have $g_{1n}^*F_1=\sum_{i=1}^n(g_{in}^{-1})_*F_i+\text{(others)}$ and,

(∗) $F_i\,(1\leq i\leq m)$ is obtained as a valuation by the weighted blow-up of X with its weights $(x,y,z)=(1,i,i)$ for local parameters x,y,z at P such that $Z_{m-1}\subseteq(g_{0,m-1}^{-1})_*\mathrm{div}(y)\cap(g_{0,m-1}^{-1})_*\mathrm{div}(z)$.

We divide the proof according to the value of $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2\leq 2$.

Case 1. $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2=0$.
This is the case when $Z_1\subseteq F_1$ is neither a line nor a point.

Case 2. $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2=1$.
This is the case when $Z_1\subseteq F_1$ is a line.

Case 3. $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2=2$.
This is the case when $Z_1\subseteq F_1$ is a point.

Since

$$\begin{aligned}\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2&=\dim_k\mathrm{Im}[(v\in\mathfrak{m}_P|Z_1\subseteq(g_1^{-1})_*\mathrm{div}(v))\to\mathfrak{m}_P/\mathfrak{m}_P^2]\\&=\dim_k\{v\in\Gamma(F_1,\mathcal{O}_{F_1}(1))|v=0\text{ or }Z_1\subseteq\mathrm{div}(v)\},\end{aligned}$$

the value of $\dim_k f_*\mathcal{O}_Y(-2E)/\mathfrak{m}_P^2$ decides the type of $Z_1\subseteq F_1\cong\mathbb{P}_k^2$ as above.

In Case 1, $\sum_{Q\in I}\min\{v_Q,r_Q-v_Q\}=0$ by (D). Therefore I is empty and thus Y is Gorenstein. By [Cu88, Theorem 5], f must be the blow-up of X along P, that is, $f=g_1$, and so we have nothing to do. Thus we have only to consider Cases 2 and 3. In these cases we investigate the values of $\dim_k\mathcal{O}_X/f_*\mathcal{O}_Y(-iE)$'s carefully.

Proposition 2.8. *(Notation as above). Let $2 \leq l \leq n$ be an integer such that $g_{0i*}(-iF_i) \not\subseteq \mathfrak{m}_P^2$ for any $i < l$.*

(1) If $g_{0l*}(-lF_l) \not\subseteq \mathfrak{m}_P^2$, then

$$\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-lE) \leq l - \frac{1}{2}\min_{0\leq j<l}\{((1+j)m-2l)j\}.$$

(2) If $g_{0l*}(-lF_l) \subseteq \mathfrak{m}_P^2$ (in this case we have $l > m$ by $(*)$), then

$$\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-lE) > l - \frac{1}{2}\min_{0\leq j<l}\{((1+j)m-2l)j\}.$$

Remark 2.8.1. In the case when $m = 1$ because

$$\min_{0\leq j<l}\{((1+j)m-2l)j\} = \min_{0\leq j<l}\{(j-(2l-1))j\} = -l(l-1),$$

we can simplify the above inequalities:

(1) $\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-lE) \leq \dfrac{1}{2}l(l+1)$.

(2) $\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-lE) > \dfrac{1}{2}l(l+1)$.

Proof. (1) By the assumption and $f_*\mathcal{O}_Y(-2E) \neq \mathfrak{m}_P$, the proof of Proposition 2.4 says that we can take local parameters x, y, z at P such that $Z_{\min\{l,m\}-1} \subseteq (g_{0,\min\{l,m\}-1}^{-1})_*\mathrm{div}(y)$ and $Z_{l-1} \subseteq (g_{0,l-1}^{-1})_*\mathrm{div}(z)$. Then for $1 \leq i \leq l$, F_i equals, as valuations, the exceptional divisor obtained by the weighted blow-up of X with its weights $(x, y, z) = (1, \min\{i, m\}, i)$.

Hence

$$\begin{aligned} f_*\mathcal{O}_Y(-lE) &= g_{0n*}\mathcal{O}_{X_n}(-lF_n) \\ &\supseteq g_{0l*}\mathcal{O}_{X_l}(-lF_l) = (x^s y^t z^u | s + \min\{l,m\}t + lu \geq l), \end{aligned}$$

and so

$$\begin{aligned} \dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-lE) &\leq \dim_k \mathcal{O}_X/(x^s y^t z^u | s + \min\{l,m\}t + lu \geq l) \\ &= l - \frac{1}{2}\min_{0\leq j<l}\{((1+j)m-2l)j\}. \end{aligned}$$

Here we used Lemma 2.9 proved later.

(2) As in the proof of (1), we can take local parameters x, y, z at P such that $Z_{m-1} \subseteq (g_{0,m-1}^{-1})_*\mathrm{div}(y)$ and $Z_{l-2} \subseteq (g_{0,l-2}^{-1})_*\mathrm{div}(z)$. Then for $1 \leq i < l$, F_i equals, as valuations, the exceptional divisor obtained by the weighted blow-up of X with its weights $(x, y, z) = (1, \min\{i, m\}, i)$.

We have

$$\begin{aligned} f_*\mathcal{O}_Y(-lE) &= g_{0n*}\mathcal{O}_{X_n}(-lF_n) \\ &\subseteq g_{0,l-1*}\mathcal{O}_{X_{l-1}}(-lF_{l-1}) + (v \in \mathfrak{m}_P | Z_{l-1} \subseteq (g_{0,l-1}^{-1})_*\mathrm{div}(v)). \end{aligned}$$

But since

$$(v \in \mathfrak{m}_P | Z_{l-1} \subseteq (g_{0,l-1}^{-1})_* \mathrm{div}(v)) \subseteq g_{0l*}\mathcal{O}_{X_l}(-lF_l) \subseteq \mathfrak{m}_P^2,$$

for any $v \in \mathfrak{m}_P$ such that $Z_{l-1} \subseteq (g_{0,l-1}^{-1})_* \mathrm{div}(v)$ we have

$$g_{0,l-1}^* \mathrm{div}(v) \geq g_{1,l-1}^*(2F_1 + (g_1^{-1})_* \mathrm{div}(v)) \geq (2 + (l-2))F_{l-1} = lF_{l-1}.$$

Thus

$$(v \in \mathfrak{m}_P | Z_{l-1} \subseteq (g_{0,l-1}^{-1})_* \mathrm{div}(v)) \subseteq g_{0,l-1*}\mathcal{O}_{X_{l-1}}(-lF_{l-1}),$$

and hence

$$f_*\mathcal{O}_Y(-lE) \subseteq g_{0,l-1*}\mathcal{O}_{X_{l-1}}(-lF_{l-1}) = (x^s y^t z^u | s + mt + (l-1)u \geq l).$$

Therefore with Lemma 2.9,

$$\begin{aligned}
\dim_k \mathcal{O}_X / f_*\mathcal{O}_Y(-lE) &\geq \dim_k \mathcal{O}_X/(x^s y^t z^u | s + mt + (l-1)u \geq l) \\
&> \dim_k \mathcal{O}_X/(x^s y^t z^u | s + mt + lu \geq l) \\
&= l - \frac{1}{2} \min_{0 \leq j < l} \{((1+j)m - 2l)j\}.
\end{aligned}$$

□

We used the following lemma in the above proof.

Lemma 2.9. *Let $X \ni P$ be an algebraic germ of a smooth 3-dimensional variety with local parameters x, y, z at P, and let l, m be positive integers. Then*

$$\dim_k \mathcal{O}_X/(x^s y^t z^u | s + \min\{l, m\}t + lu \geq l) = l - \frac{1}{2} \min_{0 \leq j < l} \{((1+j)m - 2l)j\}.$$

Proof.

$$\begin{aligned}
&\dim_k \mathcal{O}_X/(x^s y^t z^u | s + \min\{l,m\}t + lu \geq l) \\
&= \dim_k \mathrm{Span}_k \langle x^s y^t | s + \min\{l,m\}t < l \rangle \\
&= \sum_{0 \leq t \leq \lfloor \frac{l}{\min\{l,m\}} \rfloor} (l - \min\{l,m\}t) \\
&= \sum_{0 \leq t \leq \lfloor \frac{l}{m} \rfloor} (l - mt) \\
&= l - \frac{m}{2} \left\{ \left(\left\lfloor \frac{l}{m} \right\rfloor + \frac{1}{2} - \frac{l}{m} \right)^2 - \left(\frac{1}{2} - \frac{l}{m} \right)^2 \right\} \\
&= l - \frac{m}{2} \min_{0 \leq j < l} \left\{ \left(j + \frac{1}{2} - \frac{l}{m} \right)^2 - \left(\frac{1}{2} - \frac{l}{m} \right)^2 \right\} \\
&= l - \frac{1}{2} \min_{0 \leq j < l} \{((1+j)m - 2l)j\}.
\end{aligned}$$

□

Now we prove $f_*\mathcal{O}_Y(-nE) \not\subseteq \mathfrak{m}_P^2$ in Cases 2 and 3.

Proof in Case 2. For $Z_1 \subseteq F_1$ is a line in this case and a is the discrepancy of F_n with respect to K_X, we get $m = 1$ and

$$a = n + 1 \quad (n \geq 2). \tag{2.10}$$

By (D), $\sum_{Q\in I} \min\{v_Q, r_Q - v_Q\} = 1$ and thus $\{(r_Q, \overline{v_Q})\}_{Q\in I} = \{(r, \overline{\pm 1})\}$. From (B), we obtain $aE^3 = (r + 1)/r$. By (A),

$$a \leq r + 1. \tag{2.11}$$

From (C), Remark 2.7.2, and (2.10), for $1 \leq i \leq n + 1$ we have

$$\begin{aligned}\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) &= i^2 - \frac{1}{2}\min_{0\leq j<i}\{(1 + j)jr + i(i - 1 - 2j)\}\\ &= \frac{1}{2}i(i + 1) - \frac{1}{2}\min_{0\leq j<i}\{((1 + j)r - 2i)j\}.\end{aligned}$$

Hence for $1 \leq i \leq n + 1$,

$$\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) \geq \frac{1}{2}i(i + 1), \tag{2.12}$$

where the equality holds if and only if $i \leq r$.

If there exists a positive integer $2 \leq l \leq n$ such that $g_{0l*}\mathcal{O}_{X_l}(-lF_l) \subseteq \mathfrak{m}_P^2$ and $g_{0i*}\mathcal{O}_{X_i}(-iF_i) \not\subseteq \mathfrak{m}_P^2$ for any $i < l$, then by Proposition 2.8, Remark 2.8.1, and the condition of the equality in (2.12), we obtain $l = r + 1$. Thus with (2.10), we have $r + 1 = l \leq n = a - 1$, that is, $a \geq r + 2$. This contradicts (2.11) and hence we get $g_{0n*}\mathcal{O}_{X_n}(-nF_n) \not\subseteq \mathfrak{m}_P^2$. ☐

Proof in Case 3. In this case we use essentially the same idea as in Case 2, but it is a little more complicated. By (D), $\sum_{Q\in I} \min\{v_Q, r_Q - v_Q\} = 2$. Thus we have only to consider the two subcases:

Subcase 1. $\{(r_Q, \overline{v_Q})\}_{Q\in I} = \{(r, \overline{\pm 2})\}$, $r \geq 5$.

Subcase 2. $\{(r_Q, \overline{v_Q})\}_{Q\in I} = \{(r_1, \overline{\pm 1}), (r_2, \overline{\pm 1})\}$, $2 \leq r_1 \leq r_2$.

In Subcase 1, we have $aE^3 = 4/r$ by (B) and thus $a \leq 4$ from (A). But since $Z_1 \subseteq F_1$ is a point, we get $n = 2$ and $a = 4$. Then choosing local parameters x, y, z at P such that $Z_1 \subseteq (g_1^{-1})_*\mathrm{div}(y) \cap (g_1^{-1})_*\mathrm{div}(z)$, F_2 equals, as valuations, the exceptional divisor obtained by the weighted blow-up of X with its weights $(x, y, z) = (1, 2, 2)$. So we have only to investigate Subcase 2.

Recalling that a is the discrepancy of F_n with respect to K_X, we have

$$a = m + n \quad (2 \leq m \leq n). \tag{2.13}$$

Calculating with (B) we obtain $aE^3 = (r_1 + r_2)/r_1r_2$, and thus by (A),

$$a \leq r_1 + r_2. \tag{2.14}$$

From (C), Remark 2.7.2, and (2.13), for $1 \leq i \leq m + n$ we have

$$\begin{aligned}
&\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) \\
&= i^2 - \frac{1}{2}\min_{0\leq j<i}\{(1+j)jr_1 + i(i-1-2j)\} \\
&\quad - \frac{1}{2}\min_{0\leq j<i}\{(1+j)jr_2 + i(i-1-2j)\} \\
&= i - \frac{1}{2}\Big(\min_{0\leq j<i}\{((1+j)r_1 - 2i)j\} + \min_{0\leq j<i}\{((1+j)r_2 - 2i)j\}\Big).
\end{aligned}$$

Hence for $1 \leq i \leq m + n$,

$$\begin{aligned}
\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-iE) &\geq i - \frac{1}{2}\min_{0\leq j<i}\{((1+j)r_1 - 2i)j\} \\
&\geq i,
\end{aligned} \tag{2.15}$$

where the equality of the first inequality holds if and only if $i \leq r_2$, and the second holds if and only if $i \leq r_1$.

Claim 2.10. $r_1 = m$.

Proof of the claim. Utilizing Proposition 2.8 (1) with $l = m$, we have

$$\dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-mE) \leq m - \frac{1}{2}\min_{0\leq j<m}\{(j-1)jm\} = m. \tag{2.16}$$

We take local parameters x, y, z at P as in $(*)$, satisfying $Z_m \subseteq (g_{0m}^{-1})_*\mathrm{div}(z)$ if $Z_m \subseteq F_m \cong \mathbb{P}_k^2$ is a line. We have

$$\begin{aligned}
f_*\mathcal{O}_Y(-(m+1)E) &= g_{0n*}\mathcal{O}_{X_n}(-(m+1)F_n) \\
&\subseteq g_{0m*}\mathcal{O}_{X_m}(-(m+1)F_m) + (v \in \mathfrak{m}_P | Z_m \subseteq (g_{0m}^{-1})_*\mathrm{div}(v)).
\end{aligned}$$

But since

$$(v \in \mathfrak{m}_P | Z_m \subseteq (g_{0m}^{-1})_*\mathrm{div}(v)) \subseteq g_{0m*}\mathcal{O}_{X_m}(-mF_m) = (x^m, y, z),$$

we get

$$(v \in \mathfrak{m}_P | Z_m \subseteq (g_{0m}^{-1})_*\mathrm{div}(v)) \subseteq (z) + g_{0m*}\mathcal{O}_{X_m}(-(m+1)F_m),$$

and thus

$$\begin{aligned}
f_*\mathcal{O}_Y(-(m+1)E) &\subseteq (z) + g_{0m*}\mathcal{O}_{X_m}(-(m+1)F_m) \\
&= (z) + (x^s y^t z^u | s + mt + mu \geq m + 1).
\end{aligned}$$

Hence,

$$
\begin{aligned}
(2.17) \qquad & \dim_k \mathcal{O}_X/f_*\mathcal{O}_Y(-(m+1)E) \\
&\geq \dim_k \mathcal{O}_X/((z)+(x^s y^t z^u | s+mt+mu \geq m+1)) \\
&= \dim_k \mathrm{Span}_k\langle x^s,\, y | s \leq m\rangle \\
&= m+2.
\end{aligned}
$$

From (2.16), (2.17), and the condition of the second equality in (2.15), we have $r_1 = m$. □

If there exists a positive integer $l \leq n$ such that $g_{0l*}\mathcal{O}_{X_l}(-lF_l) \subseteq \mathfrak{m}_P^2$ and $g_{0i*}\mathcal{O}_{X_i}(-iF_i) \not\subseteq \mathfrak{m}_P^2$ for any $i < l$, then by Proposition 2.8, Claim 2.10, and the condition of the first equality in (2.15), we obtain $l = r_2 + 1$. Thus with (2.13) and Claim 2.10, we have $r_1 + r_2 + 1 = m + l \leq m + n = a$. This contradicts (2.14) and hence we get $g_{0n*}\mathcal{O}_{X_n}(-nF_n) \not\subseteq \mathfrak{m}_P^2$. □

References

[Co95] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, Jour. Alg. Geom. 4 (1995), 223-254.

[Co99] A. Corti, *Singularities of linear systems and 3-fold birational geometry*, Warwick, preprint, (1999).

[CPR99] A. Corti, A. Pukhlikov and M. Reid, *Fano 3-fold hypersurfaces*, Warwick, preprint, (1999).

[Cu88] S. Cutkosky, *Elementary contractions of Gorenstein threefolds*, Math. Ann. 280 (1988), 521-525.

[K88] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. Math. 127 (1988), 93-163.

[K96] Y. Kawamata, *Divisorial contractions to 3-dimensional terminal quotient singularities*, Higher-dimensional complex varieties (Trento 1994), de Gruyter (1996), 241-246.

[KMM87] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. St. Pure Math. 10 (1987), 283-360.

[K+92] J. Kollár et al, *Flips and abundance for algebraic threefolds*, Astérisque 211 (1992).

[M82] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. 116 (1982), 133-176.

[M88] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. 1 (1988), 117-253.

[R87] M. Reid, *Young person's guide to canonical singularities*, Proc. Symp. Pure Math. 46 (1987), 345-414.

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